PQ. Recall the following fact that you proved on the first homework. If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Which of the following statements is equivalent to this fact?

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{i=0}^{\infty} a_n$ converges

(b) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=0}^{\infty} a_n$ diverges.

(c) If $\sum_{i=0}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

Thm: (The Divergence Test) If the sequence of terms in a series do not _____, then the series _____.

Q. If you are asked to determine whether a series converges or diverges, when should you use this test?

Ex 1 Does the series $\sum_{n=1}^{\infty} \cos(1/n)$ converge or diverge?

Ex 2 Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

Note (IMPORTANT) If $\lim_{n\to\infty} a_n = 0$, the divergence test tells you _____.

Q. How can we relate series to things that we already understand? Like the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?

Ex 3 Decide whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Thm: (The Integral Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with _____ terms. Suppose there is a function f and a positive integer n_0 such that the following conditions are met:

(a) f is _____ for all $x \geqslant n_0$

(b) f is _____ for all $x \geqslant n_0$

(c) _____ for all integers $n \geqslant n_0$ (ie. f interpolates $\{a_n\}$ for all $n \geqslant n_0$)

Then

- If $\int_{n_0}^{\infty} f(x)\,dx$ _____, then $\sum_{n=1}^{\infty} a_n$ _____.
- If $\int_{n_0}^{\infty} f(x)\,dx$ _____, then $\sum_{n=1}^{\infty} a_n$ _____.

Q. What is great about this theorem?

Q. What is not great about this theorem?

Def: If p is a real number, we call the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ a _____.

Q. For which values of p does the p-series converge?

Ex 4 Decide whether the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$ converges or not.

(a) Converges

(b) Diverges

(c) Neither

Q. We know that if $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = L$ for *some* L. Can we find the value of L?

Thm: (Integral Test Remainder Estimate) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with _____ terms. Suppose there is a function f and a positive integer k such that the following conditions are met:

(a) f is continuous for $x \geqslant k$

(b) f is decreasing for $x \geqslant k$

(c) $f(n) = a_n$ for all integers $n \geqslant k$.

Then if $\sum_{n=1}^{\infty} a_n$ converges, we get a remainder estimate:

Ex 5 Consider $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$.

(a) Find explicit upper and lower bounds for $R_k = \sum_{n=k+1}^{\infty} \frac{1}{1+n^2}$ in terms of k.

(b) Use the partial sum $S_{1000} = \sum_{n=1}^{1000} \frac{1}{1+n^2}$ to estimate the value of the "full" series $L = \sum_{n=1}^{\infty} \frac{1}{1+n^2}$, and give a bound for the error.

(c) Repeat (b) with S_{100000}.

Ex 6 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$.

(a) Why does the series converge?

(b) Estimate L with error $< 10^{-5}$.